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Constrained Shape Preserving Rational Quintic Fractal Interpolation Functions

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ABSTRACT
In this study, we define$C2$- rational quintic FIF with the three shape parameters and discuss the constrained nature of given set of data. The developed RQFIF is a generalized fractal form of the classical rational quintic function of the form$\frac{p(\theta)}{q(\theta)}$. The RQFIF involves the three shape parameters and a vertical scaling factor in each sub-interval. We drive the sufficient conditions on the shape parameters and the vertical scaling factor associated with C^2RQFIF for the constrained type of data.

Keyword: shape preserving interpolation, constrained data, rational quintic FIF, iterated function system .

I. INTRODUCTION

Fractal interpolation is the modern method to analyses scientific data. In some cases, the classical interpolation is not valid so we used the interpolation methods for analyzation. Classical interpolants are not valid to describe the irregular shape of data so for this we describe the irregular shapes by using the interpolation schemes. Interpolation is used in various fields like in physics, geology, economics, computer graphics etc. fractal interpolation is one of the application parts of the IFS theory which gives the new research methods in different fields. In computer graphics, fractal interpolation methods give an option for catching the data in self similarity designs at any dimension of magnification. The unsmooth items such as clouds, coastlines, woodland skyline etc are represented by the FIF.

Barnsley[6] introduced the FIFs by using the IFS theory. FIF's points generally at data which extant details at the various scales or certain degree of self-similarity. An attractor of an IFS is a graph of the approximated functions. Barnsley and Harrington[7] presented the differentiable FIF through a fixed type of boundary conditions in constructive manner. This gives the connection between the classical type functions and fractal functions. Barnsley[3] constructed the C^n FIF's from the n+1 degree polynomials which provide some algebraic twists.

The emphasis of interpolation is to develop continuous function that fits the given data set got by experimentation or sampling. In any case, to get a effective physical interpretation of basic procedure, it is essential to create interpolation arrangements that acquire firm properties from the recommended given data set.It has been found that it is difficult for classical splines to specify the boundary conditions. Malik,Maria and Zahra[10] introduced the new class of C^2rational quintic function with the three shape parameters for the curve design. Chand et al.[4][5] introduced the spline FIF in constructive way with the general boundary conditions. The advantage of this spline FIF is that the choice of differentiable C^r-FIF's of certain order can be used to describe the flexibility. Chand[5] introduced the shape preservation of data points forC^1-cubic fractal spline. Vijender, N introduced the shape preservation of the rational fractal interpolation functions without using the shape parameters to preserve the constrained data for the first time[12]. chand et al[13][14] introduced the rational cubic spline FIF involving the shape parameters for the first time in literature. Appropriate conditions on parameters of associated IFS are established so that rational cubic fractal interpolant gets the fundamental shape properties present in the given data set. Chand[13] introduced the rational cubic spline fractal interpolation over classical interpolation by preserve the shape parameters. Further development on the different type of rational fractal interpolation functions can be seen in Prasad et al.[17] used quadratic trigonometric fractal interpolation function for approximating given data. Thus, the method of approximation by smooth FIFs and their various extensions in the form of fractal interpolation surfaces, coalescence hidden variable FIFs etc have been extensively studied in the literature (see [16, 18,19,20,21]) for define the C^2 shape preserving interpolants, we have to generate the derivatives at knots by using the fixed values. From the hermite type of data, the C^2 type interpolants may not be generated. Thus, to avoid these problems we have introduced new kind of C^2 rational quintic fractal function with three free parameters. Abbas[25] presented the shape preserving piecewise rational cubic interpolation constrained data by using the spline functions. Duan[26] introduced the new class of C^2rational interpolation which is centered on the function values and constrained control of the inerpolant curves.

The aim of this paper is to establish the monotonicity conditions of the given set of data for C^2rational quintic FIF's with three free parametersand express its advantages over corresponding classical rational quintic function.

In section 2 of the paper the introduction and basics of the FIF's are reviewed. In the section 3, we discussed construction of the C^2 rational quintic fractal interpolation function. In section 4 we developed the sufficient conditions for the constrained type of data.

II. THEORY OF FRACTAL INTERPOLATION FUNCTION

Here, we discuss basics of the *FIF's* established on *IFS* theory and their approaches.

Let ω_i : $Y \rightarrow Y$,(Y,d_Y) is a complete metric space, $i \in \Lambda$ be continuous functions where

Λ := {1,2,...,$m-1$}.Then *IFS* is represented as$I$ ={ Y ;ω_i,$i\in \Lambda$}.I is known as a hyperbolic IFS if each ω_i, $i\in \Lambda$ is a contraction map(say) with the contractive factors_i.

Then ∃ natural metric called the Hausdorff metric which completes $H(Y)$where$\mathrm{H}$(Y) be set of all non-empty compact subsets of Y. The Hausdorff metric$H(Y)$ is defined by

$d_{\mathrm{H}(Y)}(A,B) = max\{D_B(A),D_A(B)\}$,

where

$D_B(\mathrm{A}) = max_{a\in A} min_{b\in B} d_Y(a,b)$.

The Hutchinson map Jon $H(Y)$associated with the IFS defined by $J(A) = \bigcup_{i=1}^{m-1} \omega_i(A)$for all $A\in H(Y)$.J is contraction map on $H(Y)$ with contractive factor $s = max\{s_i: i = 1,2,...,m-1\}$if IFS I is hyperbolic. P has a unique fixed point (say) G by using Banach fixed point theorem such that for any $A\in H(Y)$,$\lim_{n\to\infty} W^{0(n)}(A) = G$, and the limit is taken w.r.t the $H(Y)$. The fixed point G is known as a attractor or deterministic fractal corresponding to the hyperbolic *IFS*.

Let P :$\{x_1, x_2 \dots, x_m\}$ be a partition of the real compact interval I =$[x_1, x_m]$,where$x_1 < x_2 < \cdots < x_m$.

Let a set of data points$\{(x_j, f_j) \in I \times K$: $j=1,2,3\dots,m\}$be given, where $K$is compact set in R. Set $I_i = [x_i, x_{i+i}]$andL_I: $I \rightarrow I_i$, $i\in \Lambda$ be the contractive homeomorphisms such that $L_i(x_1) = t_i$, $L_i(x_m) = x_{i+1}$ for $i\in \Lambda$.

$|L_i(x) - L_i(x')| \leq l_i |x\text{-}x'| \forall x, x' \in I$

for some $0 < l_i < 1$. (2.1)

Denote $C = I \times K$, and Define$m-1$ continuous mappings F_i:$C \rightarrow T$ satisfying the

$F_i(x_1, f_1) = f_i, \; F_i(x_m, f_m)$, $i \in \Lambda$, (2.2)

$|F_i(x,y) - F_i(x,t)| \leq |\delta_i||y - t| \; \forall \; y, z \in K,$

$0 \leq | \; \delta_i | < 1$. (2.3)

Now, define the functions $\omega_i: C \rightarrow I_i \times K \; such \; that \; \omega_i(x,f) = (L_i(x), F_i(x,f)) \; \forall \; i \in \Lambda$.

Proposition2.1:
An*IFS* I^*:={*C* ;ω_i,i= *1,2,3…..m-1*} admits a unique attractor *G* such that *G* is graph of a continuous function$f':I\rightarrow K$ which obeys$f'(x_j)$=f_j, *j = 1,2,3….m.*

The above defined function f' is called fractal interpolation function corresponding to *IFS*{*I*× *K*;$\omega_i(x,f)$ = $(L_i(x), F_i(x,f))$: *i =1,...,m−1*}. The construction off' is established on the following results:

Suppose *G* ={*g : I → R* | g is continuous, $g(x_1) = f_1$and $g(x_n) = f_m$}. Then (*G*,d_v) is a complete metric space with respect to metricd_v is induced from the supreme um norm on the *C(I)*. Define the Read-Bajraktarevi´c operator *T*on (G,d_v) as

$Tg(x) = F_i\left(L_i^{-1}(x), g\left(L_i^{-1}(x)\right)\right), x \in I_i$, *i*∈ Λ. (2.4)

According to the (2.1) and (2.2), Tg is continuous on I_i =[x_i,x_{i+1}],*i*∈ Λ and at each of the points$x_2 \ldots, x_{m-1}$Further, *T* is contraction map on the complete metric (G, d_v), that is,

$d_v(Tf, Tg) = \|Tf - Tg\|_\infty \le |\delta|_\infty d_v(f,g)$(2.5)

where $|\delta|_\infty$=max{|δ_i|:*i*∈ Λ}. T possesses a unique fixed point (say) f' on *G*, i.e., $f' \in G$such that $(Tf')(x)$ = $f'(x)$∀*x*∈ *I.* According to (2.4), the *FIF*f'satisfies the following functional equation:

$$f'(x) = F_i\left(L_i^{-1}(x), f' \circ L_i^{-1}(x)\right), \quad x \in I_i, \ i \in \Lambda$$

(2.6)

The following popular *IFS* is widely held to define *FIF:*

$$\{C; \omega_i(x,f) = (L_i(x), F_i(x,f)), i = 1,2, \ldots m-1\},$$

$$where\ L_i(x) = a_i x + b_i, F_i(x,f) = \delta_i f + s_i(x)\ with\ s_i: I \rightarrow R$$

(2.7)

are appropriate continuous functions such that(2.2–2.3) are satisfied. Theδ_i is called the scaling factor of the transformation ω_i, and $\delta = (\delta_1, \delta_2, \ldots \delta_{m-1})$is the scale vector of the *IFS*. In this study, we take $s_i(x)$ as a quintic rational function.

Theorem 2.1 (Barnsley and Harrington)[3]:

Suppose {(x_j,f_j): j= 1,2,…m} be the prescribed set of the interpolation data, where $t_1, t_2, \ldots, t_m$.Let L_i(*x*)= a_i*x*+b_i i∈ J satisfy 2.1 and F_i(*x*,*f*)= $\delta_i f+$ s_i(*x*),$s_i(x) = \frac{p_i(x)}{q_i(x)}$, $p_i(x)$ *and* $q_i(x)$are suitable chosen polynomials in *x* of degree *r, s* respectively and $q_i(x) \neq 0 \forall\, x \in [x_1, x_m]$.Assume that for some*p*≥*0,p*ϵ*Z*$|\delta_i| < a_i^p, i \in \Lambda$ Let$F_{i,n}$(*x,f*)=$\frac{\delta_i f + s_i^{(n)}(x)}{a_i^n}$,$f_{1,n} = \frac{s_1^{(n)}(x_1)}{a_1^n - \delta_1}$, $f_{m,n} = \frac{s_{m-1}^{(n)}(x_m)}{a_{m-1}^n - \delta_{m-1}}$, *n=1,2…,p,* where $s_i^n(x)$ represent the nth derivative of $s_i(x)$with respect to *x*. If$F_i^n(x_m, f_{m,n}) = F_{i+1}^n(x_1, f_{1,n})$ for *i=2,3,….m-2* and *n= 1,2,…,p*, then the IFS {*C;* $(L_i(x)\ , F_i(x,f)) : i = 1,2, \ldots, m-1\}$ determines a rational FIF $\sigma \in C^p(I)$, such that σ$(L_i(x)) = \delta_i \sigma(x) + s_i(x)$ and $\sigma^{(n)}$ is the FIF determined by {*C;*$\left(L_i(x), F_{i,n}(x,f)\right) : i = 1,2 \ldots m-1\}$ for *n=1,2,…,p.*

Based on this theorem many scientists have built the *FIF's* and splines which consider as a special case of fractal spline when α_i=0 ∀*i*∈*J*. Therefor the concept of *FIF* gives inclusive range of interpolations plans varying from nowhere differentiable interpolants to infinitely differentiable interpolants such as polynomials. Since the graph *G* of *FIF*is a union of transformed duplicates of itself, i.e., *G = wi*(*G*), other name for a fractal function could be a self-referential function.

III.CONSTRUCTION OF C^2 RATIONAL QUINTIC FIF

In this study, we take C^2as a rational quintic fractal interpolation function having three free parameters. In this, we build the RQFIF σ with three shape parameters in each subinterval with the help of Theorem 2.1. Suppose {(x_j, f_j), j $\in \Lambda'$} be a given set of interpolation data for an original function σ such that $x_1 < x_2 < \cdots . < x_m$. Consider the *IFS*{*I* ×*K*,$\omega_i(x,f)$ = $(L_i(x), F_i(x,f))$: *i* =*1,...,m−1*}whereL_i(*x*)= $a_i x + b_i$ i∈ J and F_i(*x,f*)= $\delta_i f +$ s_i(*x*),$s_i(x) = \frac{p_i(x)}{q_i(x)}$where $p_i(x)$ is a quantic polynomial and $q_i(x)$ is a quadratic polynomial,

$q_i(x) \neq 0 \ \forall\, x \in [x_1, x_m]$. By using the theorem 2.1integer $|\delta_i| < a_i^p, i = 1,2 \ldots,$

$m-1$. Let$F_i^{(1)}$(*x,d*)= $\frac{\delta_i d + s_i^{(1)}(x)}{a_i^1}$ and$F_i^{(2)}$(*x,D*)= where $s_i^{(1)}(x)\ and\ s_i^{(2)}(x)$are the first and second derivatives of $s_i(x)$ respectively. F_i*(x,σ)* satisfying the following C^2-inter polatory conditions:

$$F_i(x_1, \sigma_1) = f_i, F_i(x_m, \sigma_m) = f_{i+1}, F_i^{(1)}(x_1, d_1) = d_i\ , F_i^{(1)}(x_n, d_n) = d_{i+1}, F_i^{(2)}(x_1, D_1) = D_i, F_i^{(2)}(x_n, D_n) = D_{i+1}$$

, (3.1)

whered_i represent the firstorder derivativeof*σ*w.r.t x at knot x_i. From (2.7) one can observe that our *RQFIF* σ can be written as:

$$\sigma(L_i(x)) = \delta_i \sigma(x) + s_i(x) \quad (3.2)$$

where $\sigma(L_i(x))$is the rational quintic fractal interpolation function with vertical scaling factor δ_i, where $s_i(x)$ is the rational quintic function defined as:

$s_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \theta = \frac{x-x_1}{l}$, $l = x_m - x_1, x \epsilon I$, (3.3)

$$p_i(\theta) = \sum_{i=0}^{5} (1-\theta)^{5-i} \theta^i B_i$$

$$p_i(\theta) = (1-\theta)^5 B_0 + (1-\theta)^4 \theta B_1 + (1-\theta)^3 \theta^2 B_2 + (1-\theta)^2 \theta^3 B_3 + (1-\theta)^1 \theta^4 B_4 + \theta^5 B_5$$

(3.4)

$$q_i(\theta) = \alpha_i (1-\theta)^2 + \beta_i (1-\theta)\theta + \gamma_i \theta^2 \quad (3.5)$$

and α_i, γ_i and β_i are positive shape parameters. To shield that the fractal function σ is C^2-interpolant, we impose some interpolation properties:

$$\sigma(L_i(x_1)) = f_i, \quad \sigma(L_i(x_m)) = f_{i+1}, \quad \sigma^{(1)}(L_i(x_1)) = d_i, \quad \sigma^{(1)}(L_i(x_m)) = d_{i+1}, \quad \sigma^{(2)}(L_i(x_1)) = D_i, \quad \sigma^{(2)}(L_i(x_m)) = D_{i+1}, \ i \epsilon \Lambda.$$

Put $x = x_1$ in equation (3.2) and(3.3) then we have $\theta = 0$ and $s_i(x_1) = \frac{p_i(0)}{q_i(0)}$ and

$$\sigma(L_i(x_1)) = \delta_i \sigma(x_1) + s_i(x_1)$$

$$\Rightarrow f_i = \delta_i f_1 + \frac{p_i(0)}{q_i(0)}$$

$$\Rightarrow f_i = \delta_i f_1 + \frac{B_0}{\alpha_i}$$

$$\Rightarrow B_0 = \alpha_i f_i - \delta_i f_1 \alpha_i$$

$$\Rightarrow B_0 = \alpha_i (f_i - \delta_i f_1)$$

$$\Rightarrow B_0 = \alpha_i f_{i,1}^* \quad (3.6)$$

Put $x = x_m$ in equation (3.3) then we have $\theta = 1$ and $s_i(x_m) = \frac{p_i(1)}{q_i(1)}$ and

$$\sigma(L_i(x_m)) = \delta_i \sigma(x_m) + s_i(x_m)$$

$$\Rightarrow f_{i+1} = \delta_i f_m + \frac{p_i(1)}{q_i(1)}$$

$$\Rightarrow f_{i+1} = \delta_i f_m + \frac{B_5}{\gamma_i}$$

$\Rightarrow B_5 = \gamma_i f_{i+1} - \delta_i f_m \gamma_i$

$\Rightarrow B_5 = \gamma_i (f_{i+1} - \delta_i f_m)$

$\Rightarrow B_5 = \gamma_i f^*_{i+1,m}$ (3.7)

Differentiate (3.2) to (3.5) w.r.t x then we get

$\sigma'(L_i(x))L_i'(x) = \delta_i \sigma'(x) + s_i'(x)$ (3.8)

and $s_i'(x) = \frac{q_i(\theta)p_i'(\theta) - q_i'(\theta)p_i(\theta)}{l\,(q_i(\theta))^2}$ **(3.9)**

$p_i'(\theta) = -5(1-\theta)^4 B_0 - 4(1-\theta)^3\theta B_1 + (1-\theta)^4 B_1 - 3(1-\theta)^2\theta^2 B_2 + 2(1-\theta)^3\theta^1 B_2 + 3(1-\theta)^2\theta^2 B_3 + 2(1-\theta)^1\theta^3 B_3 + 4(1-\theta)^1\theta^3 B_4 - \theta^4 B_4 + 5\theta^4 B_5$ (3.10)

$q_i'(\theta) = -2\alpha_i(1-\theta)^1 + \beta_i(1-\theta) - \beta_i\theta + 2\gamma_i\theta$ (3.11)

Put $x = x_1$ in equation (3.2) and (3.8) then we have $\theta = 0$ and

$\sigma'(L_i(x_1))L_i'(x_1) = \delta_i \sigma'(x_1) + s_i'(x_1)$

$$\Rightarrow a_i d_i = \delta_i d_1 + \frac{q_i(0)p_i'(0) - q_i'(0)p_i(0)}{l\,(q_i(0))^2}$$

$$\Rightarrow d_i a_i l = \delta_i d_1 l + \frac{\alpha_i(-5B_0 + B_1) - B_0(-2\alpha_i + \beta_i)}{\alpha_i^2}$$

$\Rightarrow d_i a_i l\alpha_i^2 = \delta_i d_1 l\alpha_i^2 + (-5B_0\alpha_i + B_1\alpha_i + 2B_0\alpha_i - \beta_i B_0)$

$\Rightarrow d_i a_i l\alpha_i^2 = \delta_i d_1 l\alpha_i^2 - 3B_0\alpha_i + B_1\alpha_i - \beta_i B_0$

Put B_0 from (3.6)

$\Rightarrow d_i a_i l\alpha_i^2 = \delta_i d_1 l\alpha_i^2 - 3\alpha_i(\alpha_i(f_i - \delta_i f_1) + B_1\alpha_i - \beta_i(\alpha_i(f_i - \delta_i f_1))$

$\Rightarrow d_i a_i l\alpha_i^2 = \delta_i d_1 l\alpha_i^2 - 3\alpha_i^2 f_i + 3\alpha_i^2\delta_i f_1 + B_1\alpha_i - \beta_i\alpha_i f_i + \beta_i\alpha_i\delta_i f_1$

eliminate α_i

$\Rightarrow d_i a_i l\alpha_i = \delta_i d_1 l\alpha_i - 3\alpha_i f_i + 3\alpha_i^1\delta_i f_1 + B_1 - \beta_i f_i + \beta_i\delta_i f_1$

$\Rightarrow B_1 = d_i a_i l\alpha_i - \delta_i d_1 l\alpha_i + 3\alpha_i f_i - 3\alpha_i^1\delta_i f_1 + B_1 + \beta_i f_i - \beta_i\delta_i f_1$

$\Rightarrow B_1 = \alpha_i l(a_i d_i - \delta_i d_1) + (3\alpha_i + \beta_i)(f_i - \delta_i f_1)$

$\Rightarrow B_1 = l\alpha_i d_{i,1}^* + (3\alpha_i + \beta_i) f_{i,1}^*$ (3.12)

Put $x = x_m$ in equation (3.3) and (3.8) then we have $\theta = 1$

$\sigma'(L_i(x_m))L_i'(x_m) = \delta_i \sigma'(x_m) + s_i'(x_m)$

$$\Rightarrow a_i d_{i+1} = \delta_i d_m + \frac{q_i(1)p_i'(1) - q_i'(1)p_i(1)}{l\,(q_i(1))^2}$$

$\Rightarrow a_i l d_{i+1}\gamma_i^2 = \delta_i d_m l \gamma_i^2 + (-B_4\gamma_i + 5B_5\gamma_i + \beta_i B_5 - 2\gamma_i B_5)$

Put B_5 from (3.7)

$$\Rightarrow a_i l d_{i+1}\gamma_i^2 = \delta_i d_m l \gamma_i^2 + (-B_4\gamma_i + 5\gamma_i(\gamma_i(f_{i+1} - \delta_i f_m) + \beta_i(\gamma_i(f_{i+1} - \delta_i f_m)) - 2\gamma_i(\gamma_i(f_{i+1} - \delta_i f_m))$$

eliminate γ_i

$\Rightarrow B_4 = \gamma_i l(-a_i d_{i+1} + \delta_i d_m) + (3\gamma_i + \beta_i)(f_{i+1} - \delta_i f_m)$

$\Rightarrow B_4 = -d_{i+1,m}^* \gamma_i l + (3\gamma_i + \beta_i) f_{i+1,m}^*$ (3.13)

Again differentiate (3.8) to (3.11) then we get

$\sigma''(L_i(x))L_i'(x) = \delta_i \sigma''(x) + s_i''(x)$ (3.14)

and $s_i''(x) = \dfrac{p_i''(\theta)(q_i(\theta))^2 - q_i''(\theta)p_i(\theta)q_i(\theta) + 2p_i(\theta)(q_i'(\theta))^2 - 2p_i'(\theta)q_i'(\theta)q_i(\theta)}{l^2\,(q_i(\theta))^3}$ **(3.15)**

$$p_i''(\theta) = 20(1-\theta)^3 B_0 + 12(1-\theta)^2\theta B_1 - 8(1-\theta)^3 B_1 - 12(1-\theta)^2\theta^1 B_2 + 6(1-\theta)^1\theta^2 B_2 + 2(1-\theta)^3 B_2 + 6(1-\theta)^2\theta^1 B_3 - 12(1-\theta)^1\theta^2 B_3 + 2\theta^3 B_3 + 12(1-\theta)^1\theta^2 B_4 - 8\theta^3 B_4 + 20\theta^3 B_5$$

(3.16)

$q_i''(\theta) = 2(\alpha_i - \beta_i + \gamma_i)$ (3.17)

put $x = x_1 \Rightarrow \theta = 0$ and equation (3.14) to (3.17) implies

$\sigma''(L_i(x_1))L_i'(x_1) = \delta_i \sigma''(x_1) + s_i''(x_1)$

$$\Rightarrow a_i^2 D_i = \delta_i D_1 + \frac{p_i''(0)(q_i(0))^2 - q_i''(0)p_i(0)q_i(0) + 2p_i(0)(q_i'(0))^2 - 2p_i'(0)q_i'(0)q_i(0)}{l^2\,(q_i(0))^3}$$

$$\Rightarrow a_i^2 D_i l^2 \alpha_i^3 = \delta_i D_1 l^2 \alpha_i^3 + \{(20B_0 - 8B_1 + 2B_2)\alpha_i^2 - (2\alpha_i - 2\beta_i + 2\gamma_i)B_0\alpha_i + 2B_o(4\alpha_i^2 + \beta_i^2 - 4\alpha_i\beta_i) - 2(-5B_0 + B_1)(-2\alpha_i + \beta_i)\alpha_i\}$$

Put $B_0 and\ B_1$ from (3.6) and (3.12) then we get

$$B_2 = 0.5\alpha_i l^2 D_i a_i^2 - 0.5\alpha_i l^2 \delta_i D_1 + f_i(3\alpha_i + 3\beta_i + \gamma_i) - f_1(3\alpha_i + 3\beta_i + \gamma_i)\delta_i + l d_i a_i(2\alpha_i + \beta_i) - l d_1(2\alpha_i + \beta_i)\delta_i$$

$$B_2 = 0.5\alpha_i l^2 D_{i,1}^* + (3\alpha_i + 3\beta_i + \gamma_i)f_{i,1}^* + l d_{i,1}^*(2\alpha_i + \beta_i) \quad (3.18)$$

Put $x = x_m \Rightarrow \theta = 1$ and equation (3.14) to (3.17) implies

$$\sigma''(L_i(x_m))L_i'(x_m) = \delta_i \sigma''(x_m) + s_i''(x_m)$$

$$\Rightarrow a_i^2 D_{i+1} = \delta_i D_m + \frac{p_i''(1)(q_i(1))^2 - q_i''(1)p_i(1)q_i(1) + 2p_i(1)(q_i'(1))^2 - 2p_i'(1)q_i'(1)q_i(1)}{l^2\,(q_i(1))^3}$$

$$\Rightarrow a_i^2 D_{i+1} l^2 \gamma_i^3 = \delta_i D_1 l^2 \alpha_i^3 + \{(2B_3 - 8B_4 + 20B_5)\gamma_i^2 - (2\alpha_i - 2\beta_i + 2\gamma_i)B_5\gamma_i + 2B_5(4\gamma_i^2 + \beta_i^2 - 4\gamma_i\beta_i) - 2(5B_5 - B_4)(-2\gamma_i + \beta_i)\gamma_i\}$$

Put $B_4 and\ B_5$ from (3.7) and (3.13) then we get

$$\Rightarrow B_3 = 0.5\gamma_i l^2 D_{i+1} a_i^2 - 0.5\gamma_i l^2 \delta_i D_n + f_{i+1}(\alpha_i + 3\beta_i + 3\gamma_i) - f_m(\alpha_i + 3\beta_i + 3\gamma_i)\delta_i - l d_m(2\gamma_i + \beta_i)\delta_i) - l d_{i+1} a_i(2\gamma_i + \beta_i)$$

$$\Rightarrow B_3 = 0.5\gamma_i l^2 D_{i+1} a_i^2 - 0.5\gamma_i l^2 \delta_i D_n + (f_{i+1} - \delta_i f_m)(\alpha_i + 3\beta_i + 3\gamma_i) - l(2\gamma_i + \beta_i)(\delta_i d_m + d_{i+1} a_i)$$

$$\Rightarrow B_3 = 0.5\gamma_i l^2 D_{i+1,m}^* + f_{i+1,m}^*(\alpha_i + 3\beta_i + 3\gamma_i) - l(2\gamma_i + \beta_i)d_{i+1,m}^* \quad (3.19)$$

From (3.6), (3.7), (3.12), (3.13), (3.18) and (3.19) we get

$$B_0 = \alpha_i f_{i,1}^*$$

$$B_1 = l\alpha_i d_{i,1}^* + (3\alpha_i + \beta_i)f_{i,1}^*$$

$$B_2 = 0.5\alpha_i l^2 D_{i,1}^* + (3\alpha_i + 3\beta_i + \gamma_i)f_{i,1}^* + l d_{i,1}^*(2\alpha_i + \beta_i)$$

$$B_3 = 0.5\gamma_i l^2 D_{i+1,m}^* + f_{i+1,m}^*(\alpha_i + 3\beta_i + 3\gamma_i) - l(2\gamma_i + \beta_i)d_{i+1,m}^*$$

$$B_4 = -d_{i+1,m}^* \gamma_i l + (3\gamma_i + \beta_i)f_{i+1,m}^*$$

$$B_5 = \gamma_i f_{i+1,m}^*$$

Put the values of B_0, B_1, B_2, B_3, B_4 *and* B_5 in (3.4) then we get

$$p_i(\theta) = (1-\theta)^5\alpha_i f^*_{i,1} + (1-\theta)^4\theta\{l\alpha_i d^*_{i,1} + (3\alpha_i+\beta_i)f^*_{i,1}\} + (1-\theta)^3\theta^2\{0.5\alpha_i l^2 D^*_{i,1} + (3\alpha_i + 3\beta_i + \gamma_i)f^*_{i,1} + l d^*_{i,1}(2\alpha_i + \beta_i)\} + (1-\theta)^2\theta^3\{0.5\gamma_i l^2 D^*_{i+1,m} + f^*_{i+1,m}(\alpha_i + 3\beta_i + 3\gamma_i) - l(2\gamma_i + \beta_i)d^*_{i+1,m}\} + (1-\theta)^1\theta^4\{-a_i d^*_{i+1,m}\gamma_i l + (3\gamma_i+\beta_i)f^*_{i+1,m}\} + \theta^5\gamma_i f^*_{i+1,m}$$

In most of the cases, the derivatives are not given so they must be calculated by the some numerical data or from the given data. In this, we use the arithmetic mean method to approximate these values[2]

Remark 3.1

If$\delta_i = 0$, the RQFIF becomes the classical rational quintic function.

$Z(x) = \frac{p_i(t)}{q_i(t)}$where

$$p_i(t) = (1-t)^5 B'_0 + (1-t)^4\theta B'_1 + (1-t)^3 t^2 B'_2 + (1-t)^2 t^3 B'_3 + (1-t)^1 t^4 B'_4 + t^5 B'_5$$

$$q_i(t) = \alpha_i(1-t)^2 + \beta_i(1-t)t + \gamma_i t^2,$$

$t = \frac{x-x_i}{x_{i+1}-x_i}, x \in [x_i, x_{i+1}]$

with

$B'_0 = \alpha_i f_i,$

$B'_1 = l\alpha_i d_i + (3\alpha_i+\beta_i)f_i$

$B'_2 = 0.5\alpha_i l^2 D_i + f_i(3\alpha_i + 3\beta_i + \gamma_i) + l d_i a_i(2\alpha_i + \beta_i)$

$B'_3 = 0.5\gamma_i l^2 D_{i+1} + f_{i+1}(\alpha_i + 3\beta_i + 3\gamma_i) - l(2\gamma_i + \beta_i)d_{i+1}a_i$

$B'_4 = -d_{i+1}\gamma_i l + (3\gamma_i+\beta_i)f_{i+1}$

$B'_5 = \gamma_i f_{i+1}$

IV. SUFFICIENT CONDITIONS FOR CONSTRAINED C^2- RATIONAL QUINTIC FRACTAL INTERPOLATION FUNCTIONS

In this section, we discuss the build a constrained FIF whose graph lies above the straight line L for the positive interpolation data. Let a set of data points$\{(x_j, f_j) : j=1,2,3\ldots,m\}$ be given and lie above the straight line L, where $f_j > 0 \; \forall j$. Because of the arbitrary choice of the IFS parameters, a RQFIF may not be lie above the straight line L. to preserve the shape of the constrained data , we derive sufficient conditions on the three shape parameters and the vertical scaling factor. In this, we have presented the sufficient conditions for the constrained C^2- rational quintic fractal interpolation functions.

Theorem 4.1 Let σ be the RQFIF (3.2) defined over the interval $[x_1, x_m]$ for given data $\{(x_j, f_j) : j=1,2,3\ldots,m\}$. Let the data lie above the straight line $L : z = mx+c$ where $f_j > 0\ \forall j = 1,2,\ldots m$ and $z(L_i(x)) = \tau_i(1-\theta) + \omega_i\theta$ is the parametric form of L on $[x_1, x_m]$. Then the RQFIF σ lies above the straight line L if the following conditions are hold $\forall i \in \Lambda$:

1. The scaling factors are chosen s.t

$$\delta_i \in \begin{Bmatrix} [0, \varepsilon_i]\ if\ \varepsilon_i \in a_i^2 \\ [0, a_i^2)\ if\ a_i^2 \in \varepsilon_i \end{Bmatrix} \quad (4.1)$$

where

$$\varepsilon_i = \min\left\{a_i^2, \frac{\alpha_i f_i - \tau_i}{\alpha_i f_1}, \frac{6f_i - \omega_i}{6f_1}, \frac{6f_{i+1,m} - \tau_i}{6f_m}, \frac{\gamma_i f_{i+1} - \omega_i}{\gamma_i f_m},\right\}, \tau_i = mx_i + c\ and\ \omega_i = mx_{i+1} + c,$$

2. The shape parameters are chosen s.t

$$\begin{Bmatrix} \alpha_i \geq \frac{l d^*_{i+1,m} \Xi^*}{f^*_{i+1,m} - \omega_i} & if\ D^*_{i+1,m} \geq 0, d^*_{i+1,m} > 0 \\ \alpha_i \geq \frac{-l^2 \gamma_i D^*_{i+1,m}}{2(f^*_{i+1,m} - \omega_i)} & if\ D^*_{i+1,m} < 0, d^*_{i+1,m} < 0 \\ \alpha_i \geq 0 & otherwise \end{Bmatrix} \quad (4.2)$$

where $\Xi^* = \max\{1, \Xi\}$ if $D^*_{i+1,m} \geq 0$, where $\Xi = (\gamma_i + \beta_i)$,

$$\begin{Bmatrix} \beta_i \geq \frac{-l\alpha_i d^*_{i,1}}{(f^*_{i,1} - \tau_i)} & if\ d^*_{i,1} < 0 \\ \beta_i \geq \frac{l\gamma_i d^*_{i+1,m}}{(f^*_{i+1,m} - \omega_i)} & if\ d^*_{i+1,m} \geq 0 \\ \beta_i \geq 0 & otherwise \end{Bmatrix} \quad (4.3)$$

and

$$\begin{Bmatrix} \gamma_i > \frac{-l d^*_{i,1} \pounds^*}{2(f^*_{i,1} - \tau_i)} & if\ d^*_{i,1} < 0, D^*_{i,1} \geq 0 \\ \gamma_i \geq \frac{-l^2 \alpha_i D^*_{i,1}}{2(f^*_{i,1} - \tau_i)} & \text{if } D^*_{i,1} < 0, d^*_{i,1} \geq 0 \\ \gamma_i \geq 0 & otherwise \end{Bmatrix} \quad (4.4)$$

where $\pounds^* = \max\{1, \pounds\}$ if $D^*_{i,1} \geq 0$ where $\pounds = (\alpha_i + \beta_i)$.

Proof: Let a set of data points$\{(x_j, f_j)$: $j=1,2,3\ldots,m\}$be given and lie above the straight line $L : z = mx+c$, $x \in I$, i.e.,

$f_i \geq mx_i + c \;\forall\, i = 1,2,\ldots m.$

Now parametric form of straight line L id defined as:

$$z(L_i(x)) = m(L_i(x)) + c = \tau_i(1-\theta) + \omega_i\theta, \qquad (4.5)$$

where $L_i(x) = a_i x + b_i$ with $a_i = \frac{x_{i+1}-x_i}{x_m-x_1}$ and $b_i = \frac{x_n x_i - x_1 x_{i+1}}{x_m - x_1}, \theta = \frac{x-x_1}{l}$, $l = x_m - x_1$.At $x = x_1, \tau_i = z_i = mx_i + c$ and at $x = x_m, \omega_i = z_{i+1} = mx_{i+1} + c$.Thus the given RQFIF lies above the straight line L if

$\sigma(L_i(x)) \geq z(L_i(x)) \forall\, x \in [x_1, x_m].$

From (3.2) and (4.5), in parametric form is expressed as:

$$\delta_i\sigma(x) + \frac{p_i(\theta)}{q_i(\theta)} \geq \tau_i(1-\theta) + \omega_i\theta,$$

$$\Rightarrow \delta_i\sigma(x) + \frac{p_i(\theta)}{q_i(\theta)} - \tau_i(1-\theta) + \omega_i\theta \geq 0,$$

$$\Rightarrow \delta_i\sigma(x) + \frac{p_i^*(\theta)}{q_i(\theta)} \geq 0, \qquad (4.6)$$

$$p_i^*(\theta) = (1-\theta)^5 B_0^* + (1-\theta)^4\theta B_1^* + (1-\theta)^3\theta^2 B_2^* + (1-\theta)^2\theta^3 B_3^* + (1-\theta)^1\theta^4 B_4^* + \theta^5 B_5^*,$$

with

$B_0^* = B_o - \tau_i$

$B_1^* = B_1 - 5\tau_i - \omega_i\alpha_i + 3\alpha_i\tau_i - \beta_i\tau_i$

$B_2^* = B_2 - 10\tau_i - 2\omega_i\alpha_i + 9\alpha_i\tau_i - 2\beta_i\tau_i - \omega_i\beta_i - \tau_i\gamma_i$

$B_3^* = B_3 - 10\omega_i - 2\gamma_i\tau_i + 9\alpha_i\omega_i - 2\beta_i\omega_i - \omega_i\alpha_i - \tau_i\beta_i$

$B_4^* = B_4 - 5\omega_i + 3\gamma_i\omega_i - \omega_i\beta_i - \tau_i\gamma_i$

$B_5^* = B_5 - \omega_i$

Clearly the shape parameters $\alpha_i, \beta_i, \gamma_i \geq 0$ provide the denominator in (4.2) is positive. Thus the RQFIF preserves the constrained aspects if the numerator is positive i.e $p_i^*(\theta) \geq 0$.

$p_i^*(\theta) \geq 0$ if $B_0^* \geq 0, B_1^* \geq 0, B_2^* \geq 0, B_3^* \geq 0, B_4^* \geq 0, and\ B_5^* \geq 0, i \in \Lambda$.

Now

$$B_0^* = B_o - \tau_i = \alpha_i f_{i,1}^* - \tau_i = \alpha_i f_i - \alpha_i \delta_i f_1 - \tau_i \geq 0\ iff\ \delta_i \leq \frac{\alpha_i f_i - \tau_i}{\alpha_i f_1}.$$

Similarly,

$$B_5^* = B_5 - \omega_i = \gamma_i f_{i+1,m}^* - \omega_i = \gamma_i f_{i+1} - \gamma_i \delta_i f_m - \omega_i \geq 0\ iff\ \delta_i \leq \frac{\gamma_i f_{i+1} - \omega_i}{\gamma_i f_m}.$$

Now consider,

$$B_1^* = B_1 - 5\tau_i - \omega_i \alpha_i + 3\alpha_i \tau_i - \beta_i \tau_i$$

$$B_1^* = (f_{i,1}^* - \tau_i)(-3\alpha_i + \beta_i) + (6\alpha_i f_{i,1}^* - \alpha_i \omega_i) - 5\tau_i + l\alpha_i d_{i,1}^*$$

If $d_{i,1}^* \geq 0$ then the arbitrary $\alpha_i, \beta_i, \geq 0$ and $(6\alpha_i f_{i,1}^* - \alpha_i \omega_i) \geq 0 \Rightarrow \delta_i \leq \frac{6f_i - \omega_i}{6f_1}$ provide $B_1^* \geq 0$ as $\delta_i \leq \frac{\alpha_i f_i - \tau_i}{\alpha_i f_1}$. Otherwise we can choose $\beta_i \geq \frac{-l\alpha_i d_{i,1}^*}{(f_{i,1}^* - \tau_i)}$ for validity of $B_1^* \geq 0$.

Similarly consider

$$B_4^* = B_4 - 5\omega_i + 3\gamma_i \omega_i - \omega_i \beta_i - \tau_i \gamma_i$$

$$B_4^* = (f_{i+1,m}^* - \omega_i)(-3\gamma_i + \beta_i) + (6\gamma_i f_{i+1,m}^* - \gamma_i \tau_i) - 5\omega_i - l\gamma_i d_{i+1,m}^*$$

If $d_{i+1,m}^* \leq 0$ then the arbitrary $\gamma_i, \beta_i \geq 0$ and $(6\gamma_i f_{i+1,m}^* - \gamma_i \tau_i) \geq 0 \Rightarrow \delta_i \leq \frac{6f_{i+1,m} - \tau_i}{6f_m}$ provide $B_4^* \geq 0$ as $\delta_i \leq \frac{\gamma_i f_{i+1} - \omega_i}{\gamma_i f_m}$. Otherwise we can choose $\beta_i \geq \frac{l\gamma_i d_{i+1,m}^*}{(f_{i+1,m}^* - \omega_i)}$ for validity of $B_4^* \geq 0$.

Now take

$$B_2^* = B_2 - 10\tau_i - 2\omega_i \alpha_i + 9\alpha_i \tau_i - 2\beta_i \tau_i - \omega_i \beta_i - \tau_i \gamma_i$$

$$B_2^* = (f_{i,1}^* - \omega_i + l d_{i,1}^*)(2\alpha_i + \beta_i) + (f_{i,1}^* - \tau_i)(\alpha_i + 2\beta_i + \gamma_i) - 10\tau_i + 10\alpha_i \tau_i + \frac{l^2}{2}\alpha_i D_{i,1}^*$$

If $d_{i,1}^* \geq 0$ and $D_{i,1}^* \geq 0$ then any choice of $\alpha_i, \beta_i, \geq 0$ and $\gamma_i \geq 0$ gives $B_2^* \geq 0$ due to earlier assumptions on δ_i.

If $d^*_{i,1} \geq 0$ and $D^*_{i.1} < 0$ then we can choose $\gamma_i \geq \frac{-l^2\alpha_i D^*_{i.1}}{2(f^*_{i,1}-\tau_i)}$for validity of $B^*_2 \geq 0$ as

$\beta_i \geq \frac{-l\alpha_i d^*_{i,1}}{(f^*_{i,1}-\tau_i)}$.

We can write

$$B^*_2 = B^*_1 + (\beta_i f^*_{i,1} + 6\alpha_i\tau_i - 5\tau_i - \omega_i(\alpha_i + \beta_i)) + ((f^*_{i,1} - \tau_i)(\beta_i + \gamma_i) + ld^*_{i,1}(\alpha_i + \beta_i)) + \frac{l^2}{2}\alpha_i D^*_{i.1}$$

Since for $d^*_{i,1} < 0$, $B^*_1 \geq 0$ for $\beta_i \geq \frac{-l\alpha_i d^*_{i,1}}{(f^*_{i,1}-\tau_i)}$, for $B^*_2 \geq 0$,

1. Choose $\gamma_i > \frac{-ld^*_{i,1}£^*}{2(f^*_{i,1}-\tau_i)}$, $£^* = \max\{1, £\}$if $D^*_{i.1} \geq 0$ where $£ = (\alpha_i + \beta_i)$,

2. Choose $\gamma_i \geq \frac{-l^2\alpha_i D^*_{i.1}}{2(f^*_{i,1}-\tau_i)}$ if$D^*_{i.1} < 0$.

Finally,

$$B^*_3 = B_3 - 10\omega_i - 2\gamma_i\tau_i + 9\alpha_i\omega_i - 2\beta_i\omega_i - \omega_i\alpha_i - \tau_i\beta_i$$

$$B^*_3 = (f^*_{i+1,m} - \tau_i - ld^*_{i+1,m})(2\gamma_i + \beta_i) + (f^*_{i+1,m} - \omega_i)(\alpha_i + 2\beta_i + \gamma_i) - 10\omega_i + 10\gamma_i\omega_i + \frac{l^2}{2}\gamma_i D^*_{i+1,m}$$

If $d^*_{i+1,m} \leq 0$ and $D^*_{i+1,m} \geq 0$ then any choice of $\alpha_i, \beta_i, \geq 0$ and $\gamma_i \geq 0$ gives $B^*_3 \geq 0$ due to earlier assumptions on δ_i

If $d^*_{i+1,m} \leq 0$ and $D^*_{i+1,m} < 0$ then we can choose $\alpha_i \geq \frac{-l^2\gamma_i D^*_{i+1,m}}{2(f^*_{i+1,m}-\omega_i)}$ for validity of $B^*_3 \geq 0$ as

$\beta_i \geq \frac{l\gamma_i d^*_{i+1,m}}{(f^*_{i+1,m}-\omega_i)}$.

We can write

$$B^*_3 = B^*_4 + (\beta_i f^*_{i+1,m} + 6\gamma_i\omega_i - 5\omega_i - \tau_i(\gamma_i + \beta_i)) + ((f^*_{i+1,m} - \omega_i)(\beta_i + \alpha_i) - ld^*_{i+1,m}(\gamma_i + \beta_i)) + \frac{l^2}{2}\gamma_i D^*_{i+1,m}$$

Since for $d^*_{i+1,m} > 0$, $B^*_4 \geq 0$ for $\beta_i \geq \frac{l\gamma_i d^*_{i+1,m}}{(f^*_{i+1,m}-\omega_i)}$, for $B^*_2 \geq 0$,

1. Choose $\alpha_i \geq \frac{l d^*_{i+1,m} \Xi^*}{f^*_{i+1,m} - \omega_i}$, $\Xi^* = \max\{1, \Xi\}$ if $D^*_{i+1,m} \geq 0$, where $\Xi = (\gamma_i + \beta_i)$,

2. Choose $\alpha_i \geq \frac{-l^2 \gamma_i D^*_{i+1,m}}{2(f^*_{i+1,m} - \omega_i)}$ $if\ D^*_{i+1,m} < 0$.

Combining all the cases, thr RQFIF lies above the straight line if (4.1)-(4.4) holds $\forall i \in \Lambda$.

Corollary 4.1: If $\tau_i = 0\ and\ \omega_i = 0$, then the RQFIF σ is positive if the following conditions are satisfied $\forall i \in \Lambda$:

1. The scaling factors are chosen s.t

$$\delta_i \in \begin{Bmatrix} [0, \varepsilon_i]\ if\ \varepsilon_i \in a_i^2 \\ [0, a_i^2)\ if\ a_i^2\ \in \varepsilon_i \end{Bmatrix}$$

where $\varepsilon_i = \min\left\{a_i^2, \frac{\alpha_i f_i}{\alpha_i f_1}, \frac{6 f_i}{6 f_1}, \frac{6 f_{i+1,m}}{6 f_m}, \frac{\gamma_i f_{i+1}}{\gamma_i f_m}\right\}$

2. The shape parameters are chosen s.t

$$\begin{Bmatrix} \alpha_i \geq \frac{l d^*_{i+1,m} \Xi^*}{f^*_{i+1,m}} & if\ D^*_{i+1,m} \geq 0, d^*_{i+1,m} > 0 \\ \alpha_i \geq \frac{-l^2 \gamma_i D^*_{i+1,m}}{2 f^*_{i+1,m}} & if\ D^*_{i+1,m} < 0, d^*_{i+1,m} < 0 \\ \alpha_i \geq 0 & otherwise \end{Bmatrix} \begin{Bmatrix} \beta_i \geq \frac{-l \alpha_i d^*_{i,1}}{f^*_{i,1}} & if\ d^*_{i,1} < 0 \\ \beta_i \geq \frac{l \gamma_i d^*_{i+1,m}}{f^*_{i+1,m}} & if\ d^*_{i+1,m} \geq 0 \\ \beta_i \geq 0 & otherwise \end{Bmatrix}$$

and

$$\begin{Bmatrix} \gamma_i > \frac{-l d^*_{i,1} \mathcal{E}^*}{2 f^*_{i,1}} & if\ d^*_{i,1} < 0, D^*_{i,1} \geq 0 \\ \gamma_i \geq \frac{-l^2 \alpha_i D^*_{i,1}}{2 f^*_{i,1}} & \text{if } D^*_{i,1} < 0, d^*_{i,1} \geq 0 \\ \gamma_i \geq 0 & otherwise \end{Bmatrix}$$

V. CONCLUSION

In this study, we have constructed C^2- rational quintic fractal interpolation functions having the six shape parameters and three free parameters. The C^2- rational quintic fractal interpolation function become the classical rational quantic function if the scaling factor become zero. In this paper, we have derive the sufficient conditionson the IFS parameters to preserve the constrained data type function in such a way that the RQFIF lie above the straight line L. By perturbing the scaling factors and the shape parameters, shape of the curve can be modified according to desire. Therefore, the constructed FIF has more influence on shape preserving problem.

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