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Various Contraction Conditions in Digital Metric Spaces

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ABSTRACT

Digital Contractions are the application of digital metric fixed point theory that are useful to study the properties of digital images.The aim of this paper is to prove the existence of fixed point theorems for Hardy-Roger's and Zamfirescu contraction in setting of digital metric Space.

KEYWORDS : digital image , digital continuity , fixed points , fixed point theory , contractions

I. INTRODUCTION

Fixed point theory is a very complex and beautiful subject to study the continuation of fixed pointsf(t) = t. The knowledge of fixed points and its properties plays a very important role in many branches such as computer applications,biology, analysis and topology. Besides, it has some applications in computer graphics and image processing. There are many physical problems that can also be resolved using the perception of fixed points. The theory of fixed points begins with the celebration of Banach Contraction Principle which was introduced by S. Banach in 1922[1] which states that " Let T be a mapping from a complete metric space (X, d) into itself satisfying ; $d(px,qy) \leq \alpha d(x,y)$" .Till now, several developments have been done.fixed point theory continues to develop with the new computations and comes out with the new and improved results.

Digital Topology is an important tool to study the properties and features of 2-D and 3-D digital images using topological properties of the objects. This interesting topic has been studied by many researchers such as Rosenfeld, Han, and many more. The main aim is not only to study the similarity between Digital topology and fixed points but also to study the difference between topology and digital images. Digital topology has anemphasis on grindingtopological properties of n-D digital images [2, 3], which comprisessome fields of computer science[4, 5, 6]. To be definite, the origin of digital continuity was established by Rosenfeld [7] for revising 2-D and 3-D digital-images. Then this perception was stretched into the study of n-D digital images

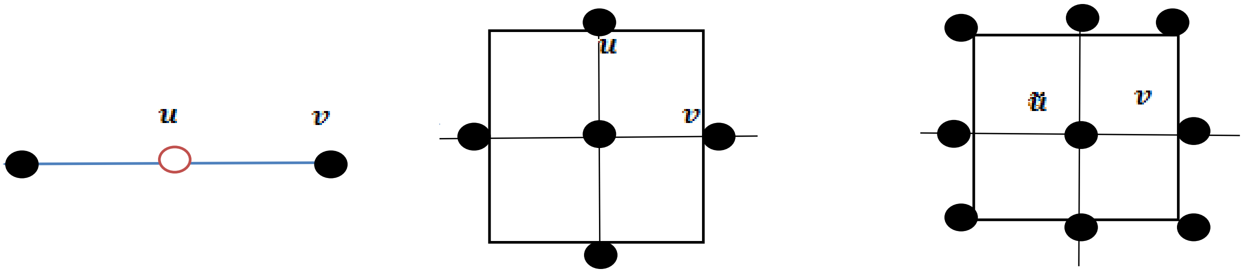
[3]. Till date now, a number ofdigital continuities have been settled [8,9,5] for grinding digital-images from the perspective of digital topology. Based on this approach, [10] firstly studied the property of digital-images.Contraction conditions of digital-images was recently studied in [11].

2.Preliminaries

Definition 2.1[3]: Let B be a subset of $\square^n$ for integer $n \geq 0$ where $\square^n$ in the Euclidean space of dimension nis the set of lattice points and ϑstands for adjacent relation for the members of B. An ordered pair(B, ϑ) is a digital-image .

Definition 2.2[3]: Two distinct points $u, v \in \mathbb{Z}^n$ are ϑ- adjacent for nD digital-images if they satisfy the below conditions

Configuration of digital connectivity



(a) 2-adjacent **(b) 4-adjacent** **(c) 8-adjacent**

For a natural number l ,$1 \leq l \leq n$, two distinct points ;$u = (u_1, u_2, u_3, \dots u_n)$

And $v = (v_1, v_2, v_3, \dots \dots .., v_n) \in \mathbb{Z}^n$

Andϑ- adjacent if atmost l of the co-ordinates differ by ± 1 and all other points coincide.

Therefore ,theϑ- adjacency relations of $\mathbb{Z}^n$ as follows :

$$\vartheta := \vartheta(l, n) = \sum_{i=n-l}^{n-1} 2^{n-i} C_i^n \; .\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(2.1)$$

Where;$C_i^n = \frac{n!}{i!(n-i)!}$

Definition 2.3[12]: Let $(B, \vartheta_0) \subset \square^{n_0}$ and $(C, \vartheta_1) \subset \square^{n_1}$ be digital-images and a function $f: B \to C$. Then f is digitally continuous at $b_0 \in B$ iff for every $\varepsilon \geq 1$ there is $\delta \geq 1$ such that $b \in B$ and $e_{k_0}(b_0, b) \leq \delta$ implies $e_{\vartheta_1}(b_1, b) \leq \varepsilon$

Definition 2.4[13]: Let $p, q \in \square^n$ with $p \leq q$, then the set

$$[p, q]_\square = \{z \in \square | p \leq z \leq q\}$$

is called 2-adjacency digital interval.

Using the ϑ-adjacency relations of $\square^n$ of (2.1), a digital neighbourhood of u in $\square^n$ is the set [16]$\square_k(u) = \{v | v \text{ is adjacent to } u\}$.

Furthermore, using the notation of [13]

$$N_\vartheta^*(u) = N_\vartheta(u) \cup \{u\}$$

Definition 2.5[14]: A digital-image $B \subset \mathbb{Z}^n$ is ϑ-connected iff each pair of altered points $u, v \in B$ there is a sequence $\{u_i\}$ where $i \in (0, l)$ of points of a digital-image B such that $u = u_0$ and $v = v_l$ with u_i and u_{i+1} are ϑ-neighbors. There is a simple ϑ path with *l* elements, length of whose elements is the number *l* denoted by $l_\vartheta(u, v)$

For a digital image (B, ϑ) as a generalization of $\square_k^*$ the digital ϑ -neighbourhood of $u_o \in B$ with radius ε is defined in B to be the following subset of B

$$N_\vartheta(u_0, \varepsilon) = \{u \in B | l_\vartheta(u_0, u) \leq \varepsilon\} \cup \{u_0\}$$

where $l_k(u_0, u)$ is the length of a shortest simple ϑ-path from u_0 to u and $\varepsilon \in \square$. Concretely, for $B \subset \square^n$ we obtain

$$N_\vartheta(u, 1) = N_\vartheta^*(u) \cap B$$

Proposition 2.1[3]: Let (B, ϑ_0) and (C, ϑ_1) be digital-images in $\square^{n_0}$ and $\square^{n_1}$ respectively. A function $f: (B, \vartheta_0) \to (C, \vartheta_1)$ is digitally (k_0, k_1)-continuous if and only if for every $u \in B, f\left(N_{\vartheta_0}(u, 1)\right) \subset f(N_{\vartheta_1}(f(u), 1)$

According to Proposition 2.1, the points $v \in \square_{\vartheta_0}(u, 1)$ is mapped into the points $f(v) \in \square_{\vartheta_1}(f(u), 1)$ which signifies that for the points u, v which are ϑ_0 $-$adjacent a $(\vartheta_0, \vartheta_1)$-continuous map *f* has the property;

$$f(u) = f(v)$$

$$or, f(v) \in N_{\vartheta_1}(f(u)) \cap Y$$

Definition 2.6[4,2] Let (B, ϑ_0) and (C, ϑ_1) be digital-images in $\square^{n_0}$ and $\square^{n_1}$ respectively. Then a map $g: B \to C$ is called a $(\vartheta_0, \vartheta_1)$-isomorphism if g is a digital $(\vartheta_0, \vartheta_1)$-continuous, bijective and further g^{-1} is (k_0, k_1)-continuous which can be denoted by $X \cong (K_0, K_1)Y$

Definition2.7[6] let (B, ϑ) be a digital-image set . let θ be a function from $(B, \vartheta) \times (B, \vartheta) \to \mathbb{Z}^n$ satisfying all the properties of metric space . The triplet (B, θ, ϑ) is called digital metric space.

3.Complete Digital Metric Space

Definition 3.1[6] A pair (B, θ, ϑ), together is a digital-metric space with ϑ-adjacency relation.

Definition 3.2[6] sequence $\{u_n\}$ of points of a digital metric space (B, θ, ϑ) is known to be a cauchy sequence if for all $\varepsilon > 0$ there exists $N \in \square$ such that for all $n, m > N$ then

$$\theta(u_n, u_m) < \varepsilon$$

Han [2] recognized the distance among two different points $u, v \in B$ in Euclidean space is greater than or equal to 1 as follows:

Proposition 3.1[2] consider two points u_i, u_j in a sequence $\{u_n\}$ of B in a digital metric space (B, θ, ϑ) such that they are ϑ -adjacent .i.e
$u_i \in N_\vartheta(u_j, 1)$ or $u_i \in N_\vartheta(u_j, 1)$ and $u_i \neq u_j$ Then they have the Euclidean distance $d(u_i, u_j) \in \{\sqrt{t} | t \in [1, l]_\square\}$

Proposition 3.2[2] A sequence $\{u_n\}$ of points of a digital metric space (B, θ, ϑ) is a Cauchy sequence if and only if there is $N \in \square$ such that for all $n, m > N$ we have

$\theta(u_n, u_m) < 1,$

$u = u_m$ for all $n, m > N$

By using the Proposition 3.1, the convergence of a sequence of a digital metric space (B, θ, ϑ) is defined as follows:

Definition 3.3[2] $\{u_n\}$ sequence of points in a digital metric space (B, θ, ϑ) assemble to a limit $L \in X$ if there is $N \in \square$ in such a way that for all n > N we have,

$u_n = L$, $i.e\ u_n = u_{n+1} = u_{n+2} = \cdots = L$

Definition 3.4[2] A digital metric space (B, θ, ϑ) is a complete digital metric space if any Cauchy sequence $\{u_n\}$ of points of (B, θ, ϑ) converges to a point $L \in B$

Definition 3.5[6]let(A, θ, ϑ) be any digital metric space. A self -map f on a digital metric space is said to be a digital contraction if there exists a $\in [0,1)$; such that for all, $y \in B$,$\theta(f(u), f(v)) \leq \lambda\, \theta(u, v)$.

Preposition 3.3[6]every digital contraction map $f: (B, \theta, \vartheta) \to (B, \theta, \vartheta)$ is digitally continuous.

4.Main Result:

Below, we validate fixed point theorems using contractions in digital metric space(DMS).

Theorem4.1[Zamfirescu Contraction] Let (B, θ, ϑ)be a complete digital metric space and T:$B \to B$ be an injective mapping satisfying the condition

1. $\theta(Tu, Tv) \leq \alpha\theta(u, v)$
2. $\theta(Tu, Tv) \leq \beta[\theta(u, Tu) + \theta(v, Tv)]$
3. $\theta(Tu, Tv) \leq \gamma[\theta(u, Tv) + \theta(v, Tv)]$

$$\theta(Tu, Tv) < \varphi \text{Max}\left\{d(u, v), \frac{d(u,Tu)+d(v,Tv)}{2}, \frac{d(u,Tv)+d(v,Tu)}{2}\right\} \qquad \dots (1)$$

$\forall\, \vartheta \in [0,1);\ \alpha, \beta, \gamma > 0; u, v \in B \,\&\, u \neq v$,havea fixed point if $\alpha + \beta + \gamma < 1$ and moreover a unique point if $\alpha + \gamma < 1$.

ProofLet$\{u_n\}$be a sequence in B defined as $u_0 \in B$, and consider the iterate $Tu_n = u_{n+1,}$ if for some n

$Tu_n = u_{n,}$, then x_n is a fixed point.let $Tu_n \neq u_{n,}$then using the condition (1), we have

$$\theta(u_{n+1}, u_{n+2}) = \theta(Tu_n, Tu_{n+1})$$

$$\leq \varphi Max\{\theta(u_n, u_{n+1}), \left(\frac{\theta(u_n, Tu_n) + \theta(u_{n+1}, Tu_{n+1})}{2}\right), \left(\frac{\theta(u_n, Tu_{n+1}) + \theta(u_{n+1}, Tu_n)}{2}\right)\}$$

$$\theta(u_{n+1}, u) \leq \varphi Max\{\theta(u_n, u_{n+1}), \left(\frac{\theta(u_n, u_{n+1}) + \theta(u_{n+1}, u_{n+2})}{2}\right), \left(\frac{\theta(u_n, u_{n+2}) + \theta(u_{n+1}, u_{n+1})}{2}\right)\}$$

$$\theta(u_{n+1}, u_{n+2}) \leq \varphi Max\{\theta(u_n, u_{n+1}), \left(\frac{\theta(u_n, u_{n+1}) + \theta(u_{n+1}, u_{n+2})}{2}\right), \left(\frac{\theta(u_n, u_{n+1}) + \theta(u_{n+1}, u_{n+2})}{2}\right)\}$$

$$\Rightarrow \theta(u_{n+1}, u_{n+2}) \leq \varphi Max\, \theta(u_n, u_{n+1})$$

Successive iteration shows that

$d(u_{n+1},u_{n+2}) \leq \varphi\, \theta(u_n,u_{n+1}) \leq \varphi^2\, \theta(u_{n-1},u_n) \ldots \leq \varphi^n\, \theta(u_0,u_1)$

$\theta(u_{n+1},u_{n+2}) \leq \varphi^n\, \theta(u_0,u_1)$

As we know if $\{u_n\}_{n\to\infty}$ be a sequence in digital metric space (B,θ,ϑ) such that

$$\theta(u_{n+1},u_{n+2}) \leq \varphi^n\, \theta(u_0,u_1,\vartheta)$$

$\forall\, k \in [0,1) \&\ n = 1,2,3, \ldots$ then $\{u_n\}_{n\to\infty}$ is a Cauchy sequence in (B,θ,ϑ). Since (B,θ,ϑ) is a complete digital metric space; $\{u_n\}_{n\to\infty}$ converses. Let $u \in B$, then $lim_{n\to\infty} u_n \to u^*$. Again T is continuous, therefore

$Tu^* = T(lim_{n\to\infty} u_n) = \ lim_{n\to\infty} Tu_n = u^* \Rightarrow Tu^* = u^*$

Implies T has a fixed point $Tu^* = u^*$ in B.

Now we will show that u^* is unique. for that suppose v^* is another fixed point therefore $Tv^* = v^*$. Therefore by inequality (1) we have

$\theta(Tu^*,Tv^*) \leq \varphi Max\{\ \theta(u^*,v^*), \left(\frac{\theta(u^*,Tv^*)+\theta(v^*,Tv^*)}{2}\right), \left(\frac{\theta(u^*,Tv^*)+\theta(v^*,Tu^*)}{2}\right)\}$

$\theta(u^*,v^*) \leq \varphi\theta(u^*,v^*)$

$\Rightarrow (1-\varphi)d(u^*,v^*) \leq 0$

$\Rightarrow d(u^*,v^*) = 0$ since, $\varphi > 1 \Rightarrow u^* = v^*$ implies that u, v^* are not two different point but are same. Hence T has a unique point.

Theorem 4.2[Hardy AndRogers contraction] Let (B,θ,ϑ) be a complete digital metric space and $T: B \to B$ be an injective mapping satisfying the condition

$(Tu,Tv) \leq \alpha_1\, \theta(u,v) + \alpha_2\, \theta(u,Tu) + \alpha_3\, \theta(v,Tv) + \alpha_4\, \theta(u,Tv) + \alpha_5\theta(v,Tu)$...(2)

$\forall\, \vartheta \in [0,1);\ \alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 > 0; u,v \in B \,\&\, u \neq v$ havea fixed point if $\alpha_1+\alpha_2+\alpha_3+2\alpha_4 < 1$ and moreover a unique point if $\alpha_1+\alpha_4+\alpha_5 < 1$.

Proof: Let $\{u_n\}$be a sequence in B defined as $x_0 \in X$, and consider the iterate $Tu_n = u_{n+1,}$ if for some n

$Tu_n = u_{n,}$, then u_n is a fixed point. Let $Tu_n \neq u_{n,}$then using the condition (2), we have
$\theta(u_{n+1},u_{n+2}) = \theta(Tu_n,Tu_{n+1})$

$$\leq \alpha_1\theta(u_n,u_{n+1}) + \alpha_2\theta(u_n,Tu_n) + \alpha_3\theta(u_{n+1},Tu_{n+1}) + \alpha_4\theta(u_n,Tu_{n+1}) + \alpha_5\theta(u_{n+1},Tu_n)$$

$$\leq \alpha_1\theta(u_n,u_{n+1})+\alpha_2\theta(u_n,u_{n+1})+\alpha_3\theta(u_{n+1},u_{n+2})+\alpha_4\theta(u_n,u_{n+2}) + \alpha_5\theta(u_{n+1},u_{n+1})$$

$$\leq \alpha_1\theta(u_n,u_{n+1})+\alpha_2\theta(u_n,u_{n+1})+\alpha_3\theta(u_{n+1},u_{n+2})+\alpha_4[\theta(u_n,u_{n+1})+\theta(u_{n+1},u_{n+2})]$$

$$(1-\alpha_3-\alpha_4)\theta(u_{n+1},u_{n+2}) \leq (\alpha_1+\alpha_2+\alpha_4)\,\theta(u_n,u_{n+1})$$

$$\theta(u_{n+1},u_{n+2}) \leq \frac{(\alpha_1+\alpha_2+\alpha_4)}{(1-\alpha_3-\alpha_4)}\theta(u_n,u_{n+1})$$

$$\theta(u_{n+1},u_{n+2}) \leq \varphi\,\theta(u_n,u_{n+1})$$

Where $\varphi = \frac{(\alpha_1+\alpha_2+\alpha_4)}{(1-\alpha_3-\alpha_4)} < 1 \Rightarrow \alpha_1+\alpha_2+\alpha_3+2\alpha_4 < 1$

Continuing iterations up to n times

$\theta(u_{n+1},u_{n+2}) \leq \varphi^n\,\theta(u_0,u_1)$

As we know if$\{u_n\}_{n\to\infty}$be a sequence in digital metric space(B,θ,ϑ) such that $\theta(u_{n+1},u_{n+2}) \leq \varphi^n\,\theta(u_0,u_1)$

$\forall\,\vartheta \in [0,1)\&\ n = 1,2,3,\ldots$then$\{u_n\}_{n\to\infty}$ is a Cauchy sequence in (B,θ,ϑ). Since (B,θ,ϑ) is a complete digital metric space; $\{u_n\}_{n\to\infty}$converses.Let$u^* \in B$, then $lim_{n\to\infty} u_n \to u^*$. Again T is continuous, therefore

$Tu^* = T(lim_{n\to\infty} u_n) = lim_{n\to\infty} Tu_n = u^* \Rightarrow Tu^* = u^*$

Implies T_p has a fixed point$Tu^* = u^*$ in B.

Now we will show that u^* is unique.for that suppose v^* is another fixed point therefore $T_p v^* = v^*$. Therefore by inequality (1) we have

$\theta(Tu^*,Tv^*) \leq \alpha_1\theta(u^*,v^*)+\alpha_2\theta(u^*Tv^*)+\alpha_3\theta(v^*,Tv^*)+\alpha_4\theta(u^*,Tv^*)+\alpha_5\theta(v^*,Tu^*)$

$\theta(u^*,v^*) \leq \alpha_1\theta(u^*,v^*)+\alpha_2\theta(u^*,u^*)+\alpha_3\theta(v^*,v^*)+\alpha_4\theta(u^*,v^*)+\alpha_5\theta(v^*,u^*)$

$\theta(u^*,v^*) \leq \alpha_1\theta(u^*,v^*)+\alpha_4\theta(u^*,v^*)+\alpha_5\theta(u^*,v^*)$

$\theta(u^*,v^*) \leq (\alpha_1+\alpha_4+\alpha_5)\theta(u^*,v^*)$

$(1-\alpha_1-\alpha_4-\alpha_5)\theta(x^*,y^*,\vartheta) \leq 0$

$\Rightarrow u^* = v^*$Since $\alpha_1+\alpha_4+\alpha_5 < 1$ implies that u^* and $v^*$are not different point but are same.

Hence u^* is unique.

II. CONCLUSION

We have revised the background of digital images and studied the completeness of digital metric space . Further, we proved the digital version of fixed point theorems using different contraction condition in digital metric space. We hope that the above results are helpful to understand the connection between digital images and fixed point theory and the results are applicable in image processing technique.

III. References:

[1] S. Banach, *Sur less opérationsdans les ensembles abstraitsetleur applications aux equations integrates*, Fund. Math. 3(1922), 133-181.

[2] S. E. Han, *Banach fixed point theorem from the viewpoint of digital topology*, J. Nonlinear. Sci. Appl. 9(2016), 895-905.

[3] S.E Han, *Non-product property of the digital fundamental group* , inform. Sci , 171(2005),73-91.

[4] L. Boxer, *Digitally Continuous Function*, Pattern Recognition Letter, 15(1994), 833-839

[5] L. Boxer, O. Ege, I. Karaca, J. Lopez and J. Louwsma, *Digital fixed points, approximate fixed points, and universal functions,* Applied General Topology, 17(2)(2016), 159-172

[6] O. Ege, I. Karaca, *Banach fixed point theorem for digital images*, J. Nonlinear. Sci. Appl., 8(2015), 237-245.

[7] A. Rosenfeld, *Digital topology*, Amer. Math. Soc., 86(1979), 621-630

[8] L. Boxer, *Digitally Continuous Function*, Pattern Recognition Letter, 15(1994), 833-839.

[9] O. Ege and I. Karaca, *Digital homotopy fixed point theory*, ComptesRendusMathematique, 353(11)(2015), 1029-1033

[10] S.E Han , *Almost Fixed Point Property for digital spaces associated with Marcus – Wyse Topological spaces* ,J.NonlinearSci App. , 10(2017), 34-47

[11] C. Park , O. Ege, S. Kumar, D. Jain, J.R. Lee, *fixed point theorems for various contraction conditions in digital metric space* ,Journal of Computational Analysis and Applications ,26(8)(2019) , 1451-1458,.

[12] A. Rosenfeld, *Continuous functions on digital pictures*, Pattern Recognition Letter, 4(1986), 177-184.

[13] T. Y. Kong, A. Rosenfeld, *Topological Algorithms for the Digital Image Processing*, Elsevier Science, Amsterdam, (1996).

[14]M. C. Joshi, R.K. Bose, *Some Topics in Non-linear Functional Analysis*, Wiley Eastern Limited, New Delhi 110 002, (1985).
[15]T. Zamfirescu , "*Fixed point theorems in metric spaces*" , Arch. Math. (Basel)(1972) , 292-298

[16]G.E Hardy, T.D Rogers, "*A generalization of a fixed point theorems*", Reich. Can. Math. Bull. ,1973 , 201-206 , 1973